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THE CONSTRUCTION OF SUCCESSIVE APPROXIMATIONS OF THE PERTURBATION METHOD FOR SYSTEMS WITH RANDOM COEFFICIENTS*

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A successive approximation procedure is proposed for stochastic systems reducible to standard form with non-"white noise" perturbations. To a first approximation, the solution of the perturbed system converges to the solution of some averaged deterministic system, and to a second approximation it converges to the solution of some averaged diffusion equation. Higher approximations enable one to estimate the deviations from a diffusion process. The convergence interval depends on the properties of the deterministic solution of the first-approximation equation.

1. We consider systems with equations of motion reducible to the standard form

$$x^{\cdot} = \varepsilon F(t, x) + \varepsilon^2 G(t, x),\ x(0) = a \in R_n \tag{1.1}$$

Here ε is a small parameter. For a fixed x, the functions $F(t, \cdot)$ and $G(t, \cdot)$ are stochastic processes with expectations $MF(t, \cdot) = f(t, \cdot)$, $MG(t, \cdot) = g(t, \cdot)$.

Henceforth, we assume that the functions f, g are periodic or conditionally periodic in t and the means

$$\bar{F}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t, x)\, dt \tag{1.2}$$

exist uniformly in $x \in S \subset R_n$; the function $\bar{G}(x)$ is defined similarly. Other restrictions on the coefficients of system (1.1) are stated below.

So far, two special cases of system (1.1) have been considered /1, 2/.

a) $\bar{F}(x) \not\equiv 0$. Then /1/ under appropriate restrictions the solution $x(t, \varepsilon) = x_\varepsilon(\tau_1)$ of system (1.1) weakly converges /3/ as $\varepsilon \to 0$ to a deterministic process $x_0(\tau_1)$ — the solution of the equation

$$dx_0/d\tau_1 = \bar{F}(x_0),\ x_0(0) = a,\ \tau_1 = \varepsilon t \tag{1.3}$$

If the solution $x_0(\tau_1)$ of Eq.(1.3) is asymptotically stable, then the convergence $x_\varepsilon \to x_0$ is ensured for $0 \leqslant \tau_1 < \infty$ /4/. If stability is not required, $x_\varepsilon \to x_0$ for $0 \leqslant \tau_1 \leqslant T_1$, where

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T_1 is independent of ε /1, 4/.

b) $\bar{F}(x) \equiv 0$. Then /2/ under appropriate restrictions the solution $x(t, \varepsilon) = x_\varepsilon(\tau_2)$ of system (1.1) converges weakly as $\varepsilon \to 0$ to a diffusion process - the solution of the stochastic differential equation

$$dx_0 = b(x_0)d\tau_2 + \sigma(x_0)dw, \; x_0(0) = a, \; \tau_2 = \varepsilon^2 t \tag{1.4}$$

where $w(\tau_2)$ is a standard Wiener process and the coefficients b and σ are calculated by averaging some moments of the processes F, G /2, 4/. If the solution x_0 is exponentially stable /5/, then convergence is observed for $0 \leqslant \tau_2 < \infty$ /4, 6/. If stability is not assumed, then $x_\varepsilon \to x_0$ for $0 \leqslant \tau_2 \leqslant T_2$, where T_2 is independent of ε /2, 4, 6/.

The passage to the limit from (1.1) to (1.3), (1.4) has been discussed by numerous authors (for a detailed bibliography, see, e.g., /4/).

If system (1.3) is asymptotically stable (or unstable), further refinement of the solution is obviously useless, because small corrections do not change the qualitative picture of its behaviour. The construction of higher approximations is necessary in cases when $\bar{F}(x) \not\equiv 0$ but system (1.3) is stable non-asymptotically, i.e., the deterministic solution does not provide information on the evolution of the solution of the perturbed system in large time intervals.

In what follows, we denote by $x_\varepsilon(\tau)$ the solution of some perturbed system and by $x_0(\tau)$ the solution of an approximating system (for $\varepsilon \to 0$). The trajectories of the processes $x_\varepsilon(\tau)$, $x_0(\tau)$ are in the space $D_n[0, \infty)$ of functions without discontinuities of the second kind /3/. To prove weak convergence of x_ε to x_0 as $\varepsilon \to 0$, it suffices to show that for any continuous bounded function φ defined in $D_n[0, \infty)$ we have /3/

$$\Phi_\varepsilon = \mathrm{M}\varphi(x_\varepsilon(\tau)) \to \mathrm{M}\varphi(x_0(\tau)) = \Phi_0, \; \varepsilon \to 0 \tag{1.5}$$

The problem is to construct an equation that defines with prescribed accuracy the process $x_0(\tau)$, or the functional Φ_0, and to estimate the convergence of Φ to Φ_0.

2. To construct the solution, we use the asymptotic procedure of approximating the generating differential operator of system (1.1) /4, 7, 8/ combined with the method of multiscale expansions /9/. We start with the necessary definitions /4, 7/.

Let $V(\tau)$ be a stochastic process with trajectories in R_1 defined in the standard probability space /7/ (for brevity, we only indicate the dependence of the process on the argument τ, omitting the dependence on the stochastic argument). Let $\mathrm{M}_s V(\tau)$ be the conditional expectation of the process $V(\tau)$ given $s \leqslant \tau$. We assume that with probability of unity the function $V(\tau)$ is right-continuous and is non-zero only in some finite time interval $\tau \in [0, T]$; also $\sup \mathrm{M}\,|V(\tau)| < \infty$. If the process $V(\tau)$ has these properties, then $V(\tau) \in A$.

We introduce the operator L^ε and its domain of definition $D(L^\varepsilon)$ /4, 7/. We say that $V \in D(L^\varepsilon)$ and $L^\varepsilon V = Y$ if $V, Y \in A$ and

$$\lim_{\delta \to 0+} \mathrm{M}\,|\,\delta^{-1}[\mathrm{M}_\tau V(\tau + \delta) - V(\tau)] - Y(\tau)| = 0 \tag{2.1}$$

From (2.1) it follows that /4, 7/

$$M_\tau V(\tau + \delta) - V(\tau) = \int_\tau^{\tau+\delta} M_\tau L^\varepsilon V(u)\, du \tag{2.2}$$

and from (2.2) we have /4, 7/

$$\mathrm{M}_{\tau, x} V(x_\varepsilon(\theta)) - V(x) = \int_\tau^\theta \mathrm{M}_{\tau, x} L^\varepsilon V(x_\varepsilon(u))\, du \tag{2.3}$$

From system (1.3), the operator L^ε is identical with the operator

$$L^\varepsilon = L_1 = \bar{F}'(x)\partial/\partial x \tag{2.4}$$

and for system (1.4) it is identical with the generating differential operator of a Markov process

$$L^\varepsilon = L_2 = b'(x)\partial/\partial x + {}^1/_2 \operatorname{Tr} A(x)\partial^2/\partial x^2 \tag{2.5}$$

Relationships (2.3)-(2.5) suggest a technique for calculating and comparing functionals on the trajectories of the perturbed and the approximating systems.

Let $x_0(\tau)$ be some Markov process with generating operator L (note that the solution of a deterministic system may also be treated as a Markov process /5/). As shown in /7/, if for any sufficiently smooth functions with a compact support $V(\tau, x) \in D(L)$ and any $T < \infty$, where T is independent of ε, there exists a function $V^\varepsilon(\tau)$ such that for $\tau \in [0, T]$, $\varepsilon \to 0$.

$$\lim \mathrm{M} \mid V^\varepsilon(\tau) - V(\tau, x_\varepsilon(\tau))\mid = 0 \qquad (2.6)$$
$$\lim \mathrm{M} \mid L^\varepsilon V^\varepsilon(\tau) - (\partial/\partial\tau + L)V(\tau, x_\varepsilon(\tau))\mid = 0$$

and for $\varepsilon \in (0, \varepsilon]$, $\tau \in [0, T]$ the sequence $x_\varepsilon(\tau)$ is weakly compact /3/, then the process $x_\varepsilon(\tau)$ is weakly convergnet as $\varepsilon \to 0$ to the Markov process $x_0(\tau)$ with the generating operator L.

3. Using the technique developed in /8/, we construct an approximating operator L for system (1.1). Introducing a new independent variable $\tau_2 = \varepsilon^2 t$, we define on the trajectory $x(t, \varepsilon) = x_\varepsilon(\tau_2)$ of system (1.1) some sufficiently smooth function $V(\tau_1, \tau_2, x_\varepsilon)$ that vanishes outside some bounded region S_T: $\{x_\varepsilon \in S, \tau_2 \in [0, T]\}$ and is uniformly bounded in its variables inside S_T. Construct the function $V^\varepsilon(\tau_2)$ related to $V(\tau_1, \tau_2, x_\varepsilon(\tau_2))$ by the equality

$$V^\varepsilon(\tau_2) = V(\tau_1, \tau_2, x_\varepsilon) + \varepsilon V_1 + \varepsilon^2 V_2 \qquad (3.1)$$
$$(V_k = V_k(t, \tau_1, \tau_2, x_\varepsilon),\ x_\varepsilon = x_\varepsilon(\tau_2))$$

and choose the coefficients V_1 and V_2 so that condition (2.6) is satisfied.

If the function V is independent of τ_1 and satisfies the equation

$$\partial V/\partial\tau_2 + LV = 0,\ V(T_2, x) = \varphi(x) \qquad (3.2)$$

where L is the required generating operator and φ is a sufficiently smooth function, then the estimate

$$\mid \mathrm{M}_{0,a}\varphi(x_\varepsilon(T_2)) - V(0, a)\mid \to 0,\ \varepsilon \to 0 \qquad (3.3)$$

holds if conditions (2.6) are satisfied for $0 \leqslant \tau_2 \leqslant T_2 \leqslant T$ /8/. Let us construct an analogue of Eq.(3.2) for the case when V depends on $\tau_1 = \tau_2/\varepsilon$, i.e., $\partial V/\partial\tau_2 = \varepsilon^{-1}V_{\tau_1} + V_{\tau_2}$. We write the equality /8/

$$L^\varepsilon V^\varepsilon = \varepsilon^{-1}(V_{\tau_1} + V_x'F + L_tV_1) + (V_{\tau_2} + V_{1\tau_1} + V_{1x}F + V_x'G + L_tV_2) + \qquad (3.4)$$
$$\varepsilon(V_{1\tau_2} + V_{2\tau_1} + V_{2x}'F + V_{1x}'G) + \varepsilon^2 V_{2\tau_2}$$

Here and henceforth, the prime denotes the transpose; the arguments of the functions are omitted. The operator L_t is defined in the same way as L^ε /8/:

$$L_tV = \lim_{\Delta\to 0+} \Delta^{-1}[\mathrm{M}_tV(t + \Delta, \tau, x) - V(t, \tau, x)]$$

the arguments x and $\tau = \tau_1, \tau_2$ are treated as fixed parameters. Equality is understood in a weak sense of (2.1).

To determine the generating operator L, we construct the function V so that the coefficients of ε^0, ε^{-1} on the right-hand side of (3.4) vanish. We have

$$V_{\tau_1} = -(V_x'F + L_tV_1) \qquad (3.5)$$

The function V_1 is chosen so as to eliminate rapidly oscillating terms from the right-hand side of (3.5); the function V_1 should not contain terms that are secular in t. Then /8/

$$V_1 = V_x'EF \qquad (3.6)$$

$$EF = \int_t^\infty [\mathrm{M}_tF(s, x) - \mathrm{M}F(s, x)]\,ds - \int_0^t [\mathrm{M}F(s, x) - \bar{F}(x)]\,ds$$

where $\bar{F}$ is the averaging operation (1.2). Eq.(3.6) is rewritten in the form

$$V_1 = V_x'[\Phi_F(t, x) - S_F(t, x)] \qquad (3.7)$$

where Φ_F is the stochastic and S_F the deterministic component of the operator EF.

From (3.5)-(3.7), using the properties of conditional expectation /3/, we obtain /8/

$$V_{\tau_1} + \bar{F}'V_x = V_{\tau_1} + L_1V = 0 \qquad (3.8)$$

Equating the second term in (3.4) to zero, we obtain the equality

$$V_{\tau_2} = -(V_{1\tau_1} + V_{1x}'F + V_x'G + L_tV_2) \qquad (3.9)$$

and using (3.7) and (3.8) this equality is transformed to

$$\frac{\partial V_1}{\partial\tau_1} + \frac{\partial V_1}{\partial x_i}F_i = \frac{\partial^2 V}{\partial x_i\partial x_j}(\Phi_{iF} - S_{iF})(F_j - \bar{F}_j) + \frac{\partial V}{\partial x_i}\left(\frac{\partial\Phi_{iF}}{\partial x_j} - \frac{\partial S_{iF}}{\partial x_j}\right)F_j \qquad (3.10)$$

Here the indices identify the corresponding vector components; summation over repeating indices is assumed.

Eliminating rapidly oscillating terms from the right-hand side of equality (3.9), we obtain

$$V_2(t,x) = \frac{\partial^2 V}{\partial x_i \partial x_j}(EA_{ij} + EP_{ij} + EB_{ij}) + \frac{\partial V}{\partial x_i}(EK_i + EP_i + EB_i + EG_i) \tag{3.11}$$

The operator E is defined as in (3.6), all the coefficients are evaluated at the point (t, x) and

$$\begin{gathered} A_{ij} = \Phi_{iF} F_j^0, \ P_{ij} = \Phi_{iF} f_j^0 - S_{iF} F_j^0, \ \mathrm{M} P_{ij} = 0 \\ B_{ij} = -S_{iF} f_j^0, \ \ F^0 = F - f, \ f^0 = f - \bar{F} \\ K_i = \frac{\partial \Phi_{iF}}{\partial x_j} F_j^0, \quad B_i = -\frac{\partial S_{iF}}{\partial x_j} f_j \\ P_i = \frac{\partial \Phi_{iF}}{\partial x_j} f_j - \frac{\partial S_{iF}}{\partial x_j} F_j^0, \quad \mathrm{M} P_i = 0 \end{gathered} \tag{3.12}$$

After obvious algebra, we obtain

$$V_{\tau_2} + L_2 V = 0 \tag{3.13}$$

and the operator L_2 is defined by (2.5), where

$$b(x) = \bar{K}(x) + \bar{B}(x) + \bar{G}(x)$$

and by (3.12)

$$A(x) = \{a_{ij}(x)\}, \ i, j = 1, \ldots, n \tag{3.14}$$

$$a_{ij}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \int_0^\infty \mathrm{M}\,[F_i^0(s,x) F_j^0(t,x)]\,ds$$

$$\bar{K}_i(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \int_t^\infty \mathrm{M}\left[\frac{\partial F_i^0(s,x)}{\partial x_j} F_j^0(t,x)\right] ds$$

$$B_i(x) = -\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \int_0^t \frac{\partial f_i^0(s,x)}{\partial x_j} f_j(t,x)\,ds$$

which is identical with the results obtained in /2/ for $\bar{F} = 0$.
From (3.4), (3.5) and (3.9) we have

$$L^\varepsilon V^\varepsilon = \varepsilon(R_1 + \varepsilon R_2) V \tag{3.15}$$

where R_1 and R_2 may be treated as operators acting on V.
Finally, from (3.8) and (3.13) we obtain an equation for the function V. Following the idea of multiscale expansions /9/, we represent

$$V = V(\tau_1, \tau_2, x) = V(\tau_2/\varepsilon, \tau_2, x) = V_0(\tau_2, x, \varepsilon), \ \partial V_0/\partial \tau_2 = \varepsilon^{-1} V_{0\tau_1} + V_{,0\tau_2},$$

i.e.,

$$\partial V_0/\partial \tau_2 + LV_0 = 0, \ \ V_0(T_2, x, \varepsilon) = \varphi(x) \ \ L = \varepsilon^{-1} L_1 + L_2 \tag{3.16}$$

We will show that with appropriate assumptions about the coefficients of system (1.1) and the nature of the solution of Eq.(3.16), the estimate (3.3) holds for $V = V_0$ with $0 \leqslant T_2 \leqslant T$, where T is independent of ε.

4. First let us establish the conditions when (2.6) are satisfied. We will assume that the coefficients of system (1.1) can be represented in the form

$$\begin{gathered} F(t,x) = F_0(t,x)\xi(t) + f(t,x) \\ G(t,x) = G_0(t,x)\xi(t) + g(t,x) \end{gathered} \tag{4.1}$$

where $\xi(t) \in R_l$ is a stochastic process with zero mean and $F_0(t,x)$ and $G_0(t,x)$ are appropriately dimensioned deterministic matrices. The process $\xi(t)$ satisfies the following mixing conditions (*conditions A*):

$$\begin{gathered} \mathrm{M} \mid \mathrm{M}_t[[[\xi(t_1)]^0 \xi(t_2)]^0 \ldots \xi(t_n)]^0 \leqslant c\alpha_1(t_1 - t)\alpha_2(t_2 - t_1) \ldots \\ \alpha_n(t_n - t_{n-1}) \\ t \leqslant t_1 \leqslant \ldots \leqslant t_n, \ n = 1, 2, 3 \\ [\varphi]^0 = \varphi - \mathrm{M}_\varphi, \ \mathrm{M}_\xi(t) = 0 \end{gathered}$$

where c is a constant.

$$\alpha_j(t) > 0, \quad \int_0^\infty \alpha_j(u)\,du < \infty, \quad j = 1, 2, \ldots, n$$

Conditions A are satisfied, in particular, if the components of the vector $\xi(t)$ are normal Markov stationary processes or processes satisfying the strong uniform mixing condition (these cases have been considered in detail in /4, 8/).

The coefficients $U_1 = F_0, f$ and $U_2 = G_0, g$ satisfy *conditions B*:

1) The functions U_1 and U_2 together with their derivatives U_{1x}, U_{1xx}, U_{2x} are bounded and periodic or conditionally periodic in t for $t \in (-\infty, \infty)$ uniformly in $x \in S \subset R_n$ and are continuous for $x \in R_n$ and bounded for $x \in S$ uniformly in $t \in (-\infty, \infty)$.

2) The limits (1.2), (3.14) exist uniformly in $x \in S$.

If $V_2(\tau, x, \cdot) \in C_{2,4}$ uniformly in $\varepsilon \in (0, \varepsilon_1]$, then under condition A and B we obtain from (3.6), (3.11), (3.15) /8/ that for sufficiently small ε and $\tau_2, x \in S_T$

$$M \mid V_j \mid \leqslant C_j \mid V_0 \mid, \ \mathrm{M} \mid R_j V_0 \mid \leqslant K_j \mid V_0 \mid \qquad j = 1, 2 \tag{4.2}$$

where C_j and K_j are constants independent of ε, $|V_0|$ is the norm of the function $V_0(\tau_2, x, \cdot)$ in the space $H^{l/2, l}$ and $l = 4 + \alpha$.

Thus, conditions (2.6) are satisfied if for sufficiently small ε condition C is satisfied,

$$| V_0(\tau_2, x, \varepsilon) | \leqslant C, \ \tau_2, x \in S_T \tag{4.3}$$

where C is a constant independent of ε.

For $F \not\equiv 0$, condition (4.3) may be satisfied only in some special cases. To check this condition, we usually must pass to the limit as $\varepsilon \to 0$. The easiest case to check is

$$f(t, x) = P(t)\,x, \quad F(x) = \bar{P}x \tag{4.4}$$

and all the eigenvalues of the matrix $\bar{P}$ are purely imaginary. Then making the change of variables $x = \exp(P\tau_2/\varepsilon)\,z$, we reduce Eq.(3.16) to the form (3.2) for z with coefficients that are conditionally periodic in $\tau_1 = \tau_2/\varepsilon$ and allow averaging over τ_1 /9/. The solution of the averaged equation is easily seen to be bounded, which gives the estimate (4.3) for the solution of Eq.(3.16) with the operator $L_1 = (\bar{P}x, \partial/\partial x)$. The passage to the limit is possible also in other systems with a periodic structure that allow averaging /9/.

Assume that the more general condition C′ is satisfied, which is not connected with the specific structure of system (3.16). Suppose that estimate (4.2) applies and a function $v_0(\tau_2, x) \in C_{2,4}$ exists that satisfies Eq.(3.2) with the uniformly parabolic operator

$$L = \beta'(x)\partial/\partial x + {}^1/_2 \operatorname{Tr} \alpha(x)\partial^2/\partial x^2 \tag{4.5}$$

such that

$$| v_0(\tau_2, x) | \leqslant C_0, \ \tau_2, x \in S_T$$
$$\lim_{\varepsilon \to 0} | V_0(\tau_2, x, \varepsilon) - v_0(\tau_2, x) | = 0$$

Also assume that the diffusion process $x_0(\tau_2)$ corresponding to the generating operator (4.5) is regular /5/. Then, repeating the argument of /8/, we can easily show that under conditions A, B, C or C′, the function $V = V_0$ satisfies the estimate (3.3) or the estimate

$$| M_{0,a}\varphi(x_\varepsilon(T_2)) - v_0(0, a) | \leqslant C\varepsilon \tag{4.6}$$

for $0 \leqslant T_2 \leqslant T$. Here the constants C_0, C and T are independent of ε.

Remark. In expansion (3.1) we retained only terms of order ε and ε^2. Retaining terms of higher order $\varepsilon^m V_m(t, \ldots, \tau_m, x_\varepsilon), \tau_m = \varepsilon^m t, \ m \geqslant 3$, we obtain a partial differential equation for the function V_0, with derivatives $\partial^m V_0/\partial x_m$. The appearance of the higher derivatives enables us to estimate the deviation of the process from a diffusion process. Under appropriate assumptions, the estimates (2.6) and (3.3) hold in the time interval $0 \leqslant t \leqslant T/\varepsilon^m$.

5. *Examples*. 1. The dynamic stability of the upper equilibrium of a pendulum with a vibrating suspension point (Fig.1). The equation of free oscillations of the pendulum in the neighbourhood of the upper equilibrium is reduced to the form

$$\theta^{\cdot\cdot} + 2\varepsilon^2\alpha\theta^{\cdot} - [\varepsilon^2k^2 + \varepsilon w(t)]\,\theta = 0 \tag{5.1}$$

Here θ is the angle of deflection of the pendulum, $w(t) = \xi^{\cdot\cdot}(t)$ is the acceleration of the point of suspension, k^2 and α are physical parameters of the pendulum and $\varepsilon > 0$ is a small parameter. As we know, a deterministic periodic /10/ or conditionally periodic /11/ disturbance may stabilize the upper equilibrium. Consider the case when $w(t)$ is a stochastic process. Following /11/, we reduce (5.1) to standard form by making the change of variables

$$\theta = x_1 [1 + \varepsilon\xi (t)], \quad \theta_1^{\cdot} = \varepsilon [x_2 + v (t) x_1], \quad v = \xi^{\cdot} \tag{5.2}$$

Then Eq.(5.1) reduces to

$$\begin{aligned} x_1^{\cdot} &= \varepsilon x_2 (1 - \varepsilon\xi) \\ x_2^{\cdot} &= \varepsilon (k^2 - \xi w) x_1 - \varepsilon v x_2 + \varepsilon^2 (-2\alpha v + k^2\xi) x_1 + \varepsilon^2 (-2\alpha + \xi v) x_2 \end{aligned} \tag{5.3}$$

Let $\xi(t)$ be a stationary normal Markov process with zero mean and bounded variance σ_ξ^2; then $v(t)$ and $w(t)$ are also stationary normal Markov processes, and the velocity variance σ_v^2 is assumed bounded. Therefore, the right-hand sides of system (5.3) satisfy condition A and all the results of Sect.2 apply. After obvious algebra, we obtain

$$\begin{gathered} f_1 (t, x) = x_2, \quad f_2 (t, x) = -\rho x_1. \\ \rho = \sigma_v^2 - k^2 \end{gathered} \tag{5.4}$$

i.e., the coefficient $F(x)$ has the form (4.4), where the eigenvalues $p_{1,2}$ of the matrix P are defined by the condition $p^2 + \rho^2 = 0$.

Therefore, for $\sigma_v^2 > k^2, \rho > 0$, we have $p_{1,2} = \pm i\rho^{1/2}$, i.e., the eigenvalues of the matrix P lie on the imaginary axis and the system is stabilized in the time interval $0 \leqslant t \leqslant T/\varepsilon$. To estimate the stability, consider the following approximation. From (5.3), (5.4) and (3.14), we have

$$G_1 = 0, \quad G_2 = -2\alpha x_2, \quad \bar{K} (x) = \bar{B} (x) = 0$$

The elements a_{ij} of the diffusion matrix (3.14) have the form

$$a_{11} = a_{12} = a_{21} = 0, \quad a_{22} = d^2 x_1^2$$

$$d^2 = 2\int_0^\infty K_v^2 (u)\, du$$

where $K_v(u)$ is the correlation matrix of the process $v(t)$.

Thus

$$\begin{aligned} L_1 &= x_2 \frac{\partial}{\partial x_1} - \rho x_1 \frac{\partial}{\partial x_2} \\ L_2 &= -2\alpha x_2 \frac{\partial}{\partial x_2} + \frac{d^2}{2} x_1^2 \frac{\partial^2}{\partial x_2^2} \end{aligned} \tag{5.5}$$

Calculating the moments of the processes from Eq.(3.16) with the operator (5.5), we obtain that for $\rho > 0$ and $\alpha > 0$ the system is stable in the mean, but stability in the mean square is determined by the condition $2\alpha\rho > d^2$. When this condition is violated, $Mx_{1,2}^2 \to \infty$ as $t \to \infty$ and, by (5.2), $M\theta_1^2 \to \infty$. Therefore, a higher intensity of the random disturbance characterized by the parameters σ_v, d stabilizes the mean oscillation amplitude and causes instability.

2. Instability of the stationary rotation of a pendulum. The rotation of the pendulum (Fig.1) is described by the equation

$$ml^2\theta^{\cdot\cdot} - ml (g + w (t)) \sin\theta + b\theta^{\cdot} = \mathrm{M} (t)$$

Here m is the mass, l is the length of the pendulum, b is the dissipation coefficient and $\mathrm{M} (t) = \mathrm{M}_0 \sin\omega t$ is the angular momentum.

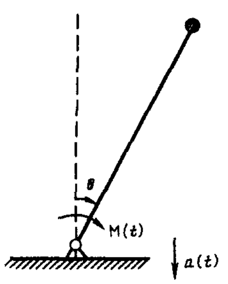

Fig.1

Introducing the small parameter $\varepsilon > 0$, we write the equation of rapid rotation of the pendulum in the form

$$\theta^{\cdot} = x, \quad \psi^{\cdot} = \omega, \quad x^{\cdot} = \varepsilon\lambda^2 [1 + \zeta (t)] \sin\theta - \varepsilon^{3/2}\beta x + \varepsilon\gamma \sin\psi$$

$$\left(\psi = \omega t, \quad \zeta = \frac{w}{g}, \quad \varepsilon\lambda^2 = \lambda_1^2 = \frac{g}{l}, \quad \varepsilon^{3/2}\beta = \beta_1 = \frac{b}{ml^2}, \quad \varepsilon\gamma = \gamma_1 = \frac{\mathrm{M}}{ml^2}\right)$$

Here we have allowed for the conditions of stationarity of rapid rotation with the frequency ω: $\lambda_1^2/\omega^2 \sim \varepsilon$, $\beta_1/\lambda_1 \sim \varepsilon$, $\gamma_1/\omega^2 \sim \varepsilon$. As in the deterministic case, let us investigate deviations of the order of $\varepsilon^{1/2}$ from the stationary rotation /12/.

Introducing new variables δ and z by the formulas

$$\theta - \psi = \delta, \quad x - \omega = \mu z, \quad \mu = \varepsilon^{1/2}$$

we obtain a standard-form equation with the small parameter μ,

$$\delta^{\cdot} = \mu z, \quad \delta (0) = \Delta \tag{5.6}$$

$$z^{\cdot} = \mu\lambda^2 [1 + \zeta(t)] \sin(\omega t + \delta) - \mu^2\beta\omega + \mu\gamma \sin \omega t + \mu^3 \ldots, \quad z(0) = \Omega$$

The averaged Eq.(1.3) for system (5.6) takes the form $\delta^{\cdot} = \mu z$, $z^{\cdot} = 0$, i.e., random perturbations do not affect the velocity of rotation. Computing the second-approximation terms, we obtain

$$\begin{gathered} B_1 = -\lambda^2\omega^{-1}z \sin\delta, \quad B_2 = 0, \quad K_1 = K_2 = 0 \\ G_1 = -\beta\omega, \quad G_2 = 0, \quad a_{12} = a_{21} = a_{22} = 0 \\ a_{11} = {}^1/_2\lambda^4 S_\zeta(\omega) = \sigma^2 \end{gathered}$$

and Eq.(3.16) takes the form

$$\frac{\partial V_0}{\partial \tau_2} + \frac{1}{\mu}\frac{\partial V_0}{\partial \delta} z - \frac{\partial V_0}{\partial z}\left(\beta\omega + \frac{\lambda}{\omega} z \sin\delta\right) + \frac{\sigma^2}{2}\frac{\partial^2 V_0}{\partial z^2} = 0 \tag{5.7}$$

The boundary conditions

$$V_0(T_2, z, \delta, \mu) = \varphi(z), \quad \tau_2 = \mu^2 t = \varepsilon t \tag{5.8}$$

characterize the change of rotation velocity.

Eq.(5.7) can be averaged over the fast variable δ /9/. The averaged equation has the form

$$\partial v_0/\partial\tau_2 - \beta\omega\partial v_0/dz + {}^1/_2\sigma^2\partial^2 v_0/\partial z^2 = 0, \quad v_0(T_2, z) = \varphi(z) \tag{5.9}$$

The generating operator of (5.9) corresponds to the diffusion process

$$dz_0 = -\beta\omega d\tau_2 + \sigma dw, \quad z_0(0) = \Omega \tag{5.10}$$

From (5.9) it follows that $Mz_0 = (\Omega - \beta\omega\tau_2)$, i.e., random disturbances do not influence the motion of the system on average. The velocity variance in turn is given by $D_z = [Mz_0^2 - (Mz_0)^2] = \sigma^2\tau_2$. Thus, without dissipation, the velocity of rotation remains constant (in the mean), but the random disturbances produce non-stationary rotation, because $D_z \to \infty$ as $t \to \infty$.

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